Mathematical Preliminaries

Exercise Set 1.1, page 14

1. For each part, $f \in C[a,b]$ on the given interval. Since $f(a)$ and $f(b)$ are of opposite sign, the Intermediate Value Theorem implies that a number c exists with $f(c) = 0$.

2. (a) $f(x) = \sqrt(x) - \cos x; f(0) = -1 < 0, f(1) = 1 - \cos 1 > 0.45 > 0$; Intermediate Value Theorem implies there is a c in $(0,1)$ such that $f(c) = 0$.
   (b) $f(x) = e^x - x^2 + 3x - 2; f(0) = -1 < 0, f(1) = e > 0$; Intermediate Value Theorem implies there is a c in $(0,1)$ such that $f(c) = 0$.
   (c) $f(x) = -3\tan(2x) + x; f(0) = 0$ so there is a c in $[0,1]$ such that $f(c) = 0$.
   (d) $f(x) = \ln x - x^2 + \frac{5}{2}x - 1; f(\frac{1}{2}) = -\ln 2 < 0, f(1) = \frac{1}{2} > 0$; Intermediate Value Theorem implies there is a c in $(\frac{1}{2}, 1)$ such that $f(c) = 0$.

3. For each part, $f \in C[a,b]$, f' exists on (a,b) and $f(a) = f(b) = 0$. Rolle's Theorem implies that a number c exists in (a,b) with $f'(c) = 0$. For part (d), we can use $[a,b] = [-1,0]$ or $[a,b] = [0,2]$.

4. (a) $[0,1]$
   (b) $[0,1]$, $[4,5]$, $[-1,0]$
   (c) $[-2,-2/3]$, $[0,1]$, $[2,4]$
   (d) $[-3,-2]$, $[-1,-0.5]$, and $[-0.5,0]$

5. The maximum value for $|f(x)|$ is given below.
   (a) 0.4620981
   (b) 0.8
   (c) 5.164000
   (d) 1.582572

6. (a) $f(x) = \frac{2x}{x^2+1}; 0 \le x \le 2; f(x) \ge 0$ on $[0,2], f'(1) = 0, f(0) = 0, f(1) = 1, f(2) = \frac{4}{5}, \max_{0 \le x \le 2} |f(x)| = 1$.
   (b) $f(x) = x^2\sqrt{4-x}; 0 \le x \le 4; f'(0) = 0, f'(3.2) = 0, f(0) = 0, f(3.2) = 9.158934436, f(4) = 0, \max_{0 \le x \le 4} |f(x)| = 9.158934436$.
   (c) $f(x) = x^3 - 4x + 2; 1 \le x \le 2; f'(\frac{2\sqrt{3}}{3}) = 0, f'(1) = -1, f(\frac{2\sqrt{3}}{3}) = -1.079201435, f(2) = 2, \max_{1 \le x \le 2} |f(x)| = 2$.

(d) $f(x) = x\sqrt{3-x^2}; 0 \le x \le 1; f'(\sqrt{\frac{3}{2}}) = 0, \sqrt{\frac{3}{2}}$ not in $[0,1], f(0) = 0, f(1) = \sqrt{2}, \max_{0\le x\le 1} |f(x)| = \sqrt{2}$.

7. For each part, $f \in C[a,b]$, f' exists on (a,b) and $f(a) = f(b) = 0$. Rolle's Theorem implies that a number c exists in (a,b) with $f'(c) = 0$. For part (d), we can use $[a,b] = [-1,0]$ or $[a,b] = [0,2]$.

8. Suppose p and q are in $[a,b]$ with $p \ne q$ and $f(p) = f(q) = 0$. By the Mean Value Theorem, there exists $\xi \in (a,b)$ with
$$f(p) - f(q) = f'(\xi)(p-q).$$
But, $f(p) - f(q) = 0$ and $p \ne q$. So $f'(\xi) = 0$, contradicting the hypothesis.

9. (a) $P_2(x) = 0$
   (b) $R_2(0.5) = 0.125$; actual error $= 0.125$
   (c) $P_2(x) = 1 + 3(x-1) + 3(x-1)^2$
   (d) $R_2(0.5) = -0.125$; actual error $= -0.125$

10. $P_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$

x	0.5	0.75	1.25	1.5
$P_3(x)$	1.2265625	1.3310547	1.5517578	1.6796875
$\sqrt{x+1}$	1.2247449	1.3228757	1.5	1.5811388
$\lvert\sqrt{x+1} - P_3(x)\rvert$	0.0018176	0.0081790	0.0517578	0.0985487

11. Since
$$P_2(x) = 1 + x \quad \text{and} \quad R_2(x) = \frac{-2e^{\xi}(\sin\xi + \cos\xi)}{6}x^3$$
for some ξ between x and 0, we have the following:
   (a) $P_2(0.5) = 1.5$ and $|f(0.5) - P_2(0.5)| \le 0.0932$;
   (b) $|f(x) - P_2(x)| \le 1.252$;
   (c) $\int_0^1 f(x)\,dx \approx 1.5$;
   (d) $|\int_0^1 f(x)\,dx - \int_0^1 P_2(x)\,dx| \le \int_0^1 |R_2(x)|\,dx \le 0.313$, and the actual error is 0.122.

12. $P_2(x) = 1.461930 + 0.617884\left(x - \frac{\pi}{6}\right) - 0.844046\left(x - \frac{\pi}{6}\right)^2$ and $R_2(x) = -\frac{1}{3}e^{\xi}(\sin\xi + \cos\xi)\left(x - \frac{\pi}{6}\right)^3$ for some ξ between x and $\frac{\pi}{6}$.
   (a) $P_2(0.5) = 1.446879$ and $f(0.5) = 1.446889$. An error bound is 1.01×10^{-5}, and the actual error is 1.0×10^{-5}.
   (b) $|f(x) - P_2(x)| \le 0.135372$ on $[0,1]$
   (c) $\int_0^1 P_2(x)\,dx = 1.376542$ and $\int_0^1 f(x)\,dx = 1.378025$
   (d) An error bound is 7.403×10^{-3}, and the actual error is 1.483×10^{-3}.

13. $P_3(x) = (x-1)^2 - \frac{1}{2}(x-1)^3$

    (a) $P_3(0.5) = 0.312500$, $f(0.5) = 0.346574$. An error bound is $0.291\overline{6}$, and the actual error is 0.034074.

    (b) $|f(x) - P_3(x)| \leq 0.291\overline{6}$ on $[0.5, 1.5]$

    (c) $\int_{0.5}^{1.5} P_3(x)\,dx = 0.08\overline{3}$, $\int_{0.5}^{1.5} (x-1)\ln x\,dx = 0.088020$

    (d) An error bound is $0.058\overline{3}$, and the actual error is 4.687×10^{-3}.

14. (a) $P_3(x) = -4 + 6x - x^2 - 4x^3$; $P_3(0.4) = -2.016$

    (b) $|R_3(0.4)| \leq 0.05849$; $|f(0.4) - P_3(0.4)| = 0.013365367$

    (c) $P_4(x) = -4 + 6x - x^2 - 4x^3$; $P_4(0.4) = -2.016$

    (d) $|R_4(0.4)| \leq 0.01366$; $|f(0.4) - P_4(0.4)| = 0.013365367$

15. $P_4(x) = x + x^3$

    (a) $|f(x) - P_4(x)| \leq 0.012405$

    (b) $\int_0^{0.4} P_4(x)\,dx = 0.0864$, $\int_0^{0.4} xe^{x^2}\,dx = 0.086755$

    (c) 8.27×10^{-4}

    (d) $P_4'(0.2) = 1.12$, $f'(0.2) = 1.124076$. The actual error is 4.076×10^{-3}.

16. First we need to convert the degree measure for the sine function to radians. We have $180° = \pi$ radians, so $1° = \frac{\pi}{180}$ radians. Since,

$$f(x) = \sin x, \quad f'(x) = \cos x, \quad f''(x) = -\sin x, \quad \text{and} \quad f'''(x) = -\cos x,$$

we have $f(0) = 0$, $f'(0) = 1$, and $f''(0) = 0$.

The approximation $\sin x \approx x$ is given by

$$f(x) \approx P_2(x) = x, \quad \text{and} \quad R_2(x) = -\frac{\cos\xi}{3!}x^3.$$

If we use the bound $|\cos\xi| \leq 1$, then

$$\left|\sin\frac{\pi}{180} - \frac{\pi}{180}\right| = \left|R_2\left(\frac{\pi}{180}\right)\right| = \left|\frac{-\cos\xi}{3!}\left(\frac{\pi}{180}\right)^3\right| \leq 8.86 \times 10^{-7}.$$

17. Since $42° = 7\pi/30$ radians, use $x_0 = \pi/4$. Then

$$\left|R_n\left(\frac{7\pi}{30}\right)\right| \leq \frac{\left(\frac{\pi}{4} - \frac{7\pi}{30}\right)^{n+1}}{(n+1)!} < \frac{(0.053)^{n+1}}{(n+1)!}.$$

For $|R_n(\frac{7\pi}{30})| < 10^{-6}$, it suffices to take $n = 3$. To 7 digits,

$$\cos 42° = 0.7431448 \quad \text{and} \quad P_3(42°) = P_3(\frac{7\pi}{30}) = 0.7431446,$$

so the actual error is 2×10^{-7}.

18. $P_n(x) = \sum_{k=0}^n x^k,\ n \geq 19$

19. $P_n(x) = \displaystyle\sum_{k=0}^{n} \frac{1}{k!} x^k,\ n \geq 7$

20. For n odd, $P_n(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots + \frac{1}{n}(-1)^{(n-1)/2}x^n$. For n even, $P_n(x) = P_{n-1}(x)$.

21. A bound for the maximum error is 0.0026.

22. For $x < 0$, $f(x) < 2x + k < 0$, provided that $x < -\frac{1}{2}k$. Similarly, for $x > 0$, $f(x) > 2x + k > 0$, provided that $x > -\frac{1}{2}k$. By Theorem 1.11, there exists a number c with $f(c) = 0$. If $f(c) = 0$ and $f(c') = 0$ for some $c' \neq c$, then by Theorem 1.7, there exists a number p between c and c' with $f'(p) = 0$. However, $f'(x) = 3x^2 + 2 > 0$ for all x.

23. Since $R_2(1) = \frac{1}{6}e^{\xi}$, for some ξ in $(0, 1)$, we have $|E - R_2(1)| = \frac{1}{6}|1 - e^{\xi}| \leq \frac{1}{6}(e - 1)$.

24. (a) Use the series

$$e^{-t^2} = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{k!} \quad \text{to integrate} \quad \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt,$$

and obtain the result.

(b) We have

$$\begin{aligned}
\frac{2}{\sqrt{\pi}} e^{-x^2} \sum_{k=0}^{\infty} \frac{2^k x^{2k+1}}{1 \cdot 3 \cdots (2k+1)} =& \frac{2}{\sqrt{\pi}} \left[1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^7 + \frac{1}{24}x^8 + \cdots\right] \\
&\cdot \left[x + \frac{2}{3}x^3 + \frac{4}{15}x^5 + \frac{8}{105}x^7 + \frac{16}{945}x^9 + \cdots\right] \\
=& \frac{2}{\sqrt{\pi}} \left[x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \frac{1}{216}x^9 + \cdots\right] = \text{erf}\ (x)
\end{aligned}$$

(c) 0.8427008

(d) 0.8427069

(e) The series in part (a) is alternating, so for any positive integer n and positive x we have the bound

$$\left| \text{erf}(x) - \frac{2}{\sqrt{\pi}} \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)k!} \right| < \frac{x^{2n+3}}{(2n+3)(n+1)!}\ .$$

We have no such bound for the positive term series in part (b).

25. (a) $P_n^{(k)}(x_0) = f^{(k)}(x_0)$ for $k = 0, 1, \ldots, n$. The shapes of P_n and f are the same at x_0.

(b) $P_2(x) = 3 + 4(x - 1) + 3(x - 1)^2$.

26. (a) The assumption is that $f(x_i) = 0$ for each $i = 0, 1, \ldots, n$. Applying Rolle's Theorem on each on the intervals $[x_i, x_{i+1}]$ implies that for each $i = 0, 1, \ldots, n - 1$ there exists a number z_i with $f'(z_i) = 0$. In addition, we have

$$a \leq x_0 < z_0 < x_1 < z_1 < \cdots < z_{n-1} < x_n \leq b.$$

(b) Apply the logic in part (a) to the function $g(x) = f'(x)$ with the number of zeros of g in $[a, b]$ reduced by 1. This implies that numbers w_i, for $i = 0, 1, \ldots, n-2$ exist with

$$g'(w_i) = f''(w_i) = 0, \quad \text{and} \quad a < z_0 < w_0 < z_1 < w_1 < \cdots w_{n-2} < z_{n-1} < b.$$

(c) Continuing by induction following the logic in parts (a) and (b) provides $n+1-j$ distinct zeros of $f^{(j)}$ in $[a, b]$.

(d) The conclusion of the theorem follows from part (c) when $j = n$, for in this case there will be (at least) $(n+1) - n = 1$ zero in $[a, b]$.

27. First observe that for $f(x) = x - \sin x$ we have $f'(x) = 1 - \cos x \geq 0$, because $-1 \leq \cos x \leq 1$ for all values of x.

(a) The observation implies that $f(x)$ is non-decreasing for all values of x, and in particular that $f(x) > f(0) = 0$ when $x > 0$. Hence for $x \geq 0$, we have $x \geq \sin x$, and $|\sin x| = \sin x \leq x = |x|$.

(b) When $x < 0$, we have $-x > 0$. Since $\sin x$ is an odd function, the fact (from part (a)) that $\sin(-x) \leq (-x)$ implies that $|\sin x| = -\sin x \leq -x = |x|$.

As a consequence, for all real numbers x we have $|\sin x| \leq |x|$.

28. (a) Let x_0 be any number in $[a, b]$. Given $\epsilon > 0$, let $\delta = \epsilon/L$. If $|x - x_0| < \delta$ and $a \leq x \leq b$, then $|f(x) - f(x_0)| \leq L|x - x_0| < \epsilon$.

(b) Using the Mean Value Theorem, we have

$$|f(x_2) - f(x_1)| = |f'(\xi)||x_2 - x_1|,$$

for some ξ between x_1 and x_2, so

$$|f(x_2) - f(x_1)| \leq L|x_2 - x_1|.$$

(c) One example is $f(x) = x^{1/3}$ on $[0, 1]$.

29. (a) The number $\frac{1}{2}(f(x_1) + f(x_2))$ is the average of $f(x_1)$ and $f(x_2)$, so it lies between these two values of f. By the Intermediate Value Theorem 1.11 there exist a number ξ between x_1 and x_2 with

$$f(\xi) = \frac{1}{2}(f(x_1) + f(x_2)) = \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2).$$

(b) Let $m = \min\{f(x_1), f(x_2)\}$ and $M = \max\{f(x_1), f(x_2)\}$. Then $m \leq f(x_1) \leq M$ and $m \leq f(x_2) \leq M$, so

$$c_1 m \leq c_1 f(x_1) \leq c_1 M \quad \text{and} \quad c_2 m \leq c_2 f(x_2) \leq c_2 M.$$

Thus

$$(c_1 + c_2)m \leq c_1 f(x_1) + c_2 f(x_2) \leq (c_1 + c_2)M$$

and

$$m \leq \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2} \leq M.$$

By the Intermediate Value Theorem 1.11 applied to the interval with endpoints x_1 and x_2, there exists a number ξ between x_1 and x_2 for which

$$f(\xi) = \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2}.$$

(c) Let $f(x) = x^2 + 1$, $x_1 = 0$, $x_2 = 1$, $c_1 = 2$, and $c_2 = -1$. Then for all values of x,

$$f(x) > 0 \quad \text{but} \quad \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2} = \frac{2(1) - 1(2)}{2 - 1} = 0.$$

30. (a) Since f is continuous at p and $f(p) \neq 0$, there exists a $\delta > 0$ with

$$|f(x) - f(p)| < \frac{|f(p)|}{2},$$

for $|x - p| < \delta$ and $a < x < b$. We restrict δ so that $[p - \delta, p + \delta]$ is a subset of $[a, b]$. Thus, for $x \in [p - \delta, p + \delta]$, we have $x \in [a, b]$. So

$$-\frac{|f(p)|}{2} < f(x) - f(p) < \frac{|f(p)|}{2} \quad \text{and} \quad f(p) - \frac{|f(p)|}{2} < f(x) < f(p) + \frac{|f(p)|}{2}.$$

If $f(p) > 0$, then

$$f(p) - \frac{|f(p)|}{2} = \frac{f(p)}{2} > 0, \quad \text{so} \quad f(x) > f(p) - \frac{|f(p)|}{2} > 0.$$

If $f(p) < 0$, then $|f(p)| = -f(p)$, and

$$f(x) < f(p) + \frac{|f(p)|}{2} = f(p) - \frac{f(p)}{2} = \frac{f(p)}{2} < 0.$$

In either case, $f(x) \neq 0$, for $x \in [p - \delta, p + \delta]$.

(b) Since f is continuous at p and $f(p) = 0$, there exists a $\delta > 0$ with

$$|f(x) - f(p)| < k, \quad \text{for} \quad |x - p| < \delta \quad \text{and} \quad a < x < b.$$

We restrict δ so that $[p - \delta, p + \delta]$ is a subset of $[a, b]$. Thus, for $x \in [p - \delta, p + \delta]$, we have

$$|f(x)| = |f(x) - f(p)| < k.$$

Exercise Set 1.2, page 28

1. We have

	Absolute error	Relative error
(a)	0.001264	4.025×10^{-4}
(b)	7.346×10^{-6}	2.338×10^{-6}
(c)	2.818×10^{-4}	1.037×10^{-4}
(d)	2.136×10^{-4}	1.510×10^{-4}

2. We have

	Absolute error	Relative error
(a)	2.647×10^{1}	1.202×10^{-3}arule
(b)	1.454×10^{1}	1.050×10^{-2}
(c)	420	1.042×10^{-2}
(d)	3.343×10^{3}	9.213×10^{-3}

3. The largest intervals are
   (a) (149.85,150.15)
   (b) (899.1, 900.9)
   (c) (1498.5, 1501.5)
   (d) (89.91,90.09)

4. The largest intervals are:
   (a) $(3.1412784, 3.1419068)$
   (b) $(2.7180100, 2.7185536)$
   (c) $(1.4140721, 1.4143549)$
   (d) $(1.9127398, 1.9131224)$

5. The calculations and their errors are:
   (a) (i) 17/15 (ii) 1.13 (iii) 1.13 (iv) both 3×10^{-3}
   (b) (i) 4/15 (ii) 0.266 (iii) 0.266 (iv) both 2.5×10^{-3}
   (c) (i) 139/660 (ii) 0.211 (iii) 0.210 (iv) 2×10^{-3}, 3×10^{-3}
   (d) (i) 301/660 (ii) 0.455 (iii) 0.456 (iv) 2×10^{-3}, 1×10^{-4}

6. We have

	Approximation	Absolute error	Relative error
(a)	134	0.079	5.90×10^{-4}
(b)	133	0.499	3.77×10^{-3}
(c)	2.00	0.327	0.195
(d)	1.67	0.003	1.79×10^{-3}

7. We have

	Approximation	Absolute error	Relative error
(a)	1.80	0.154	0.0786
(b)	-15.1	0.0546	3.60×10^{-3}
(c)	0.286	2.86×10^{-4}	10^{-3}
(d)	23.9	0.058	2.42×10^{-3}

8. We have

	Approximation	Absolute error	Relative error
(a)	1.986	0.03246	0.01662
(b)	-15.16	0.005377	3.548×10^{-4}
(c)	0.2857	1.429×10^{-5}	5×10^{-5}
(d)	23.96	1.739×10^{-3}	7.260×10^{-5}

9. We have

	Approximation	Absolute error	Relative error
(a)	3.55	1.60	0.817
(b)	-15.2	0.0454	0.00299
(c)	0.284	0.00171	0.00600
(d)	0	0.02150	1

10. We have

	Approximation	Absolute error	Relative error
(a)	1.983	0.02945	0.01508
(b)	-15.15	0.004622	3.050×10^{-4}
(c)	0.2855	2.143×10^{-4}	7.5×10^{-4}
(d)	23.94	0.018261	7.62×10^{-4}

11. We have

	Approximation	Absolute error	Relative error
(a)	3.14557613	3.983×10^{-3}	1.268×10^{-3}
(b)	3.14162103	2.838×10^{-5}	9.032×10^{-6}

12. We have

	Approximation	Absolute error	Relative error
(a)	2.7166667	0.0016152	5.9418×10^{-4}
(b)	2.718281801	2.73×10^{-8}	1.00×10^{-8}

13. (a) We have

$$\lim_{x \to 0} \frac{x \cos x - \sin x}{x - \sin x} = \lim_{x \to 0} \frac{-x \sin x}{1 - \cos x} = \lim_{x \to 0} \frac{-\sin x - x \cos x}{\sin x} = \lim_{x \to 0} \frac{-2\cos x + x \sin x}{\cos x} = -2$$

(b) $f(0.1) \approx -1.941$

(c) $\dfrac{x(1 - \frac{1}{2}x^2) - (x - \frac{1}{6}x^3)}{x - (x - \frac{1}{6}x^3)} = -2$

(d) The relative error in part (b) is 0.029. The relative error in part (c) is 0.00050.

14. (a) $\displaystyle \lim_{x \to 0} \frac{e^x - e^{-x}}{x} = \lim_{x \to 0} \frac{e^x + e^{-x}}{1} = 2$

(b) $f(0.1) \approx 2.05$

(c) $\dfrac{1}{x}\left(\left(1+x+\dfrac{1}{2}x^2+\dfrac{1}{6}x^3\right)-\left(1-x+\dfrac{1}{2}x^2-\dfrac{1}{6}x^3\right)\right)=\dfrac{1}{x}\left(2x+\dfrac{1}{3}x^3\right)=2+\dfrac{1}{3}x^2$;
using three-digit rounding arithmetic and $x = 0.1$, we obtain 2.00.

(d) The relative error in part (b) is = 0.0233. The relative error in part (c) is = 0.00166.

15.

	x_1	Absolute error	Relative error	x_2	Absolute error	Relative error
(a)	92.26	0.01542	1.672×10^{-4}	0.005419	6.273×10^{-7}	1.157×10^{-4}
(b)	0.005421	1.264×10^{-6}	2.333×10^{-4}	−92.26	4.580×10^{-3}	4.965×10^{-5}
(c)	10.98	6.875×10^{-3}	6.257×10^{-4}	0.001149	7.566×10^{-8}	6.584×10^{-5}
(d)	−0.001149	7.566×10^{-8}	6.584×10^{-5}	−10.98	6.875×10^{-3}	6.257×10^{-4}

16.

	Approximation for x_1	Absolute error	Relative error
(a)	1.903	6.53518×10^{-4}	3.43533×10^{-4}
(b)	−0.07840	8.79361×10^{-6}	1.12151×10^{-4}
(c)	1.223	1.29800×10^{-4}	1.06144×10^{-4}
(d)	6.235	1.7591×10^{-3}	2.8205×10^{-4}

	Approximation for x_2	Absolute error	Relative error
(a)	0.7430	4.04830×10^{-4}	5.44561
(b)	−4.060	3.80274×10^{-4}	9.36723×10^{-5}
(c)	−2.223	1.2977×10^{-4}	5.8393×10^{-5}
(d)	−0.3208	1.2063×10^{-4}	3.7617×10^{-4}

17.

	Approximation for x_1	Absolute error	Relative error
(a)	92.24	0.004580	4.965×10^{-5}
(b)	0.005417	2.736×10^{-6}	5.048×10^{-4}
(c)	10.98	6.875×10^{-3}	6.257×10^{-4}
(d)	−0.001149	7.566×10^{-8}	6.584×10^{-5}

	Approximation for x_2	Absolute error	Relative error
(a)	0.005418	2.373×10^{-6}	4.377×10^{-4}
(b)	-92.25	5.420×10^{-3}	5.875×10^{-5}
(c)	0.001149	7.566×10^{-8}	6.584×10^{-5}
(d)	-10.98	6.875×10^{-3}	6.257×10^{-4}

18.

	Approximation for x_1	Absolute error	Relative error
(a)	1.901	1.346×10^{-3}	7.078×10^{-4}
(b)	-0.07843	2.121×10^{-5}	2.705×10^{-4}
(c)	1.222	8.702×10^{-4}	7.116×10^{-4}
(d)	6.235	1.759×10^{-3}	2.820×10^{-4}

	Approximation for x_2	Absolute error	Relative error
(a)	0.7438	3.952×10^{-4}	5.316×10^{-4}
(b)	-4.059	6.197×10^{-4}	1.526×10^{-4}
(c)	-2.222	8.702×10^{-4}	3.915×10^{-4}
(d)	-0.3207	2.063×10^{-5}	6.433×10^{-5}

19. The machine numbers are equivalent to

    (a) 3224

    (b) -3224

    (c) 1.32421875

    (d) 1.3242187500000002220446049250313080847263336181640625

20. (a) Next Largest: 3224.00000000000045474735088646411895751953125;
    Next Smallest: 3223.99999999999954525264911353588104248046875

    (b) Next Largest: $-3224.00000000000045474735088646411895751953125$;
    Next Smallest: $-3223.99999999999954525264911353588104248046875$

    (c) Next Largest: 1.3242187500000002220446049250313080847263336181640625;
    Next Smallest: 1.3242187499999997779553950749686919152736663818359375

    (d) Next Largest: 1.324218750000000444089209850062616169452667236328125;
    Next Smallest: 1.32421875

21. (b) The first formula gives -0.00658, and the second formula gives -0.0100. The true three-digit value is -0.0116.

22. (a) -1.82

(b) 7.09×10^{-3}
(c) The formula in (b) is more accurate since subtraction is not involved.

23. The approximate solutions to the systems are
(a) $x = 2.451$, $y = -1.635$
(b) $x = 507.7$, $y = 82.00$

24. (a) $x = 2.460 \quad y = -1.634$
(b) $x = 477.0 \quad y = 76.93$

25. (a) In nested form, we have $f(x) = (((1.01e^x - 4.62)e^x - 3.11)e^x + 12.2)e^x - 1.99$.
(b) -6.79
(c) -7.07
(d) The absolute errors are

$$|-7.61 - (-6.71)| = 0.82 \quad \text{and} \quad |-7.61 - (-7.07)| = 0.54.$$

Nesting is significantly better since the relative errors are

$$\left|\frac{0.82}{-7.61}\right| = 0.108 \quad \text{and} \quad \left|\frac{0.54}{-7.61}\right| = 0.071,$$

26. Since $0.995 \le P \le 1.005$, $0.0995 \le V \le 0.1005$, $0.082055 \le R \le 0.082065$, and $0.004195 \le N \le 0.004205$, we have $287.61° \le T \le 293.42°$. Note that 15°C = 288.16K.

When P is doubled and V is halved, $1.99 \le P \le 2.01$ and $0.0497 \le V \le 0.0503$ so that $286.61° \le T \le 293.72°$. Note that 19°C = 292.16K. The laboratory figures are within an acceptable range.

27. (a) $m = 17$
(b) We have

$$\binom{m}{k} = \frac{m!}{k!(m-k)!} = \frac{m(m-1)\cdots(m-k-1)(m-k)!}{k!(m-k)!} = \left(\frac{m}{k}\right)\left(\frac{m-1}{k-1}\right)\cdots\left(\frac{m-k-1}{1}\right)$$

(c) $m = 181707$
(d) 2,597,000; actual error 1960; relative error 7.541×10^{-4}

28. When $d_{k+1} < 5$,

$$\left|\frac{y - fl(y)}{y}\right| = \frac{0.d_{k+1}\ldots \times 10^{n-k}}{0.d_1\ldots \times 10^n} \le \frac{0.5 \times 10^{-k}}{0.1} = 0.5 \times 10^{-k+1}.$$

When $d_{k+1} > 5$,

$$\left|\frac{y - fl(y)}{y}\right| = \frac{(1 - 0.d_{k+1}\ldots) \times 10^{n-k}}{0.d_1\ldots \times 10^n} < \frac{(1-0.5) \times 10^{-k}}{0.1} = 0.5 \times 10^{-k+1}.$$

29. (a) The actual error is $|f'(\xi)\epsilon|$, and the relative error is $|f'(\xi)\epsilon| \cdot |f(x_0)|^{-1}$, where the number ξ is between x_0 and $x_0 + \epsilon$.
(b) (i) 1.4×10^{-5}; 5.1×10^{-6} (ii) 2.7×10^{-6}; 3.2×10^{-6}
(c) (i) 1.2; 5.1×10^{-5} (ii) 4.2×10^{-5}; 7.8×10^{-5}

Exercise Set 1.3, page 39

1 (a) The approximate sums are 1.53 and 1.54, respectively. The actual value is 1.549. Significant roundoff error occurs earlier with the first method.

(b) The approximate sums are 1.16 and 1.19, respectively. The actual value is 1.197. Significant roundoff error occurs earlier with the first method.

2. We have

	Approximation	Absolute Error	Relative Error
(a)	2.715	3.282×10^{-3}	1.207×10^{-3}
(b)	2.716	2.282×10^{-3}	8.394×10^{-4}
(c)	2.716	2.282×10^{-3}	8.394×10^{-4}
(d)	2.718	2.818×10^{-4}	1.037×10^{-4}

3. (a) 2000 terms

   (b) 20,000,000,000 terms

4. 4 terms

5. 3 terms

6. (a) $O\left(\frac{1}{n}\right)$

   (b) $O\left(\frac{1}{n^2}\right)$

   (c) $O\left(\frac{1}{n^2}\right)$

   (d) $O\left(\frac{1}{n}\right)$

7. The rates of convergence are:

   (a) $O(h^2)$

   (b) $O(h)$

   (c) $O(h^2)$

   (d) $O(h)$

8. (a) If $|\alpha_n - \alpha|/(1/n^p) \leq K$, then

$$|\alpha_n - \alpha| \leq K(1/n^p) \leq K(1/n^q) \quad \text{since} \quad 0 < q < p.$$

Thus

$$|\alpha_n - \alpha|/(1/n^p) \leq K \quad \text{and} \quad \{\alpha_n\}_{n=1}^{\infty} \to \alpha$$

with rate of convergence $O(1/n^p)$.

(b)

n	$1/n$	$1/n^2$	$1/n^3$	$1/n^5$
5	0.2	0.04	0.008	0.0016
10	0.1	0.01	0.001	0.0001
50	0.02	0.0004	8×10^{-6}	1.6×10^{-7}
100	0.01	10^{-4}	10^{-6}	10^{-8}

The most rapid convergence rate is $O(1/n^4)$.

9. (a) If $F(h) = L + O\left(h^p\right)$, there is a constant $k > 0$ such that

$$|F(h) - L| \leq kh^p,$$

for sufficiently small $h > 0$. If $0 < q < p$ and $0 < h < 1$, then $h^q > h^p$. Thus, $kh^p < kh^q$, so

$$|F(h) - L| \leq kh^q \quad \text{and} \quad F(h) = L + O\left(h^q\right).$$

(b) For various powers of h we have the entries in the following table.

h	h^2	h^3	h^4
0.5	0.25	0.125	0.0625
0.1	0.01	0.001	0.0001
0.01	0.0001	0.00001	10^{-8}
0.001	10^{-6}	10^{-9}	10^{-12}

The most rapid convergence rate is $O\left(h^4\right)$.

10. Suppose that for sufficiently small $|x|$ we have positive constants K_1 and K_2 independent of x, for which

$$|F_1(x) - L_1| \leq K_1|x|^\alpha \quad \text{and} \quad |F_2(x) - L_2| \leq K_2|x|^\beta.$$

Let $c = \max(|c_1|, |c_2|, 1)$, $K = \max(K_1, K_2)$, and $\delta = \max(\alpha, \beta)$.

(a) We have

$$\begin{aligned}|F(x) - c_1L_1 - c_2L_2| &= |c_1(F_1(x) - L_1) + c_2(F_2(x) - L_2)| \\ &\leq |c_1|K_1|x|^\alpha + |c_2|K_2|x|^\beta \leq cK[|x|^\alpha + |x|^\beta] \\ &\leq cK|x|^\gamma[1 + |x|^{\delta-\gamma}] \leq \tilde{K}|x|^\gamma,\end{aligned}$$

for sufficiently small $|x|$ and some constant $\tilde{K}$. Thus, $F(x) = c_1L_1 + c_2L_2 + O(x^\gamma)$.

(b) We have

$$\begin{aligned}|G(x) - L_1 - L_2| &= |F_1(c_1x) + F_2(c_2x) - L_1 - L_2| \\ &\leq K_1|c_1x|^\alpha + K_2|c_2x|^\beta \leq Kc^\delta[|x|^\alpha + |x|^\beta] \\ &\leq Kc^\delta|x|^\gamma[1 + |x|^{\delta-\gamma}] \leq \tilde{K}|x|^\gamma,\end{aligned}$$

for sufficiently small $|x|$ and some constant $\tilde{K}$. Thus, $G(x) = L_1 + L_2 + O(x^\gamma)$.

11. Since
$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = x \quad \text{and} \quad x_{n+1} = 1 + \frac{1}{x_n},$$
we have
$$x = 1 + \frac{1}{x}, \quad \text{so} \quad x^2 - x - 1 = 0.$$
The quadratic formula implies that
$$x = \frac{1}{2}\left(1 + \sqrt{5}\right).$$
This number is called the *golden ratio*. It appears frequently in mathematics and the sciences.

12. Let $F_n = C^n$. Substitute into $F_{n+2} = F_n + F_{n+1}$ to obtain $C^{n+2} = C^n + C^{n+1}$ or $C^n[C^2 - C - 1] = 0$. Solving the quadratic equation $C^2 - C - 1 = 0$ gives $C = \frac{1\pm\sqrt{5}}{2}$. So $F_n = a(\frac{1+\sqrt{5}}{2})^n + b(\frac{1-\sqrt{5}}{2})$ satisfies the recurrence relation $F_{n+2} = F_n + F_{n+1}$. For $F_0 = 1$ and $F_1 = 1$ we need $a = \frac{1+\sqrt{5}}{2}\frac{1}{\sqrt{5}}$ and $b = -(\frac{1-\sqrt{5}}{2})\frac{1}{\sqrt{5}}$. Hence, $F_n = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^{n+1} - (\frac{1-\sqrt{5}}{2})^{n+1})$.

13. $SUM = \sum_{i=1}^{N} x_i$. This saves one step since initialization is $SUM = x_1$ instead of $SUM = 0$. Problems may occur if $N = 0$.

14. (a) OUTPUT is PRODUCT $= 0$ which is correct only if $x_i = 0$ for some i.
    (b) OUTPUT is PRODUCT $= x_1x_2 \dots x_N$.
    (c) OUTPUT is PRODUCT $= x_1x_2 \dots x_N$ but exists with the correct value 0 if one of $x_i = 0$.

15. (a) $n(n+1)/2$ multiplications; $(n+2)(n-1)/2$ additions.
    (b) $\displaystyle\sum_{i=1}^{n} a_i \left(\sum_{j=1}^{i} b_j\right)$ requires n multiplications; $(n+2)(n-1)/2$ additions.